JOURNAL OF APPROXIMATION THEORY 25, 105–119 (1979)

On a Class of Boundary Value Problems for a Composite System of First Order Differential Equations

R. P. GILBERT*

University of Delaware, Newark, Delaware 19711

AND

M. SCHNEIDER

Technical University Berlin, Berlin, Germany

Communicated by Oved Shisha

Received August 11, 1977

1. INTRODUCTION

Let $G \Sigma \mathbb{R}^2$ be a simply connected region surrounded by a curve $\zeta_G(s) = \{x(s), y(s)\} \in C_\alpha^1(0 \leqslant s \leqslant 1)$, $0 < \alpha < 1$. In G we consider the first order system,

$$\sum_{j=1}^{2r+1} (a_{ij}u_x^j + b_{ij}u_y^j) + \sum_{j=1}^{2r+1} c_{ij}u^j + d_i = 0, \qquad i = 1, 2,..., 2r + 1, \quad (1.1)$$

with $2r + 1$ unknowns u^j in the two independent variables x, y. The coefficients are real with

$$a_{ij}, b_{ij}(x, y) \in C_\alpha^1(\bar{G}); \qquad c_{ij}, d_i(x, y) \in C_\alpha(\bar{G});$$
$$\det(a_{ij}) \neq 0 \text{ in } G. \quad (1.2)$$

The type of (1.1) is determined by the roots of

$$\det(\mu a_{ij} - b_{ij}) = 0, \quad (1.3)$$

* This research was supported in part by the Air Force Office of Scientific Research through Grant AF-AFOSR 76-2879, and in part by the Alexander von Humboldt Foundation.

0021-9045/79/020105-15$02.00/0

and we assume that (1.3) has

$$\begin{aligned} &2r \text{ nonreal solutions:} \quad \mu_k^{(1)} = \gamma_k(x, y) + i\delta_k(x, y), \\ &\qquad\qquad\qquad\qquad\quad \mu_k^{(2)} = \gamma_k(x, y) + i\delta_k(x, y), \qquad k = 1, 2, \dots, r, \\ &\text{and one real solution} \quad \mu^{(3)} = \lambda(x, y). \end{aligned}$$

From (1.2) we have γ_k, δ_k, $\lambda \in C_\alpha^1(\bar{G})$. (1.1) is a composite system of $2r$ elliptic and one hyperbolic equation, and for $r = 1$ we get the system of Vidic [8]. The real characteristics of (1.1) in $\bar{G}$ are given by the solutions of the ordinary differential equations

$$\frac{dx}{ds} = \lambda_1(x, y), \qquad \frac{dy}{ds} = \lambda_2(x, y), \qquad \lambda_1^2 + \lambda_2^2 = 1, \tag{1.4}$$

with $dy/dx = \lambda(x, y)$. We shall suppose that these characteristics intersect the boundary ∂G in exactly two points with the exception of two curves, each of which touches ∂G in only one point, either $\zeta_G(s_1)$ or $\zeta_G(s_2)$. Let $\partial G_1 := \{\zeta_G(s) \mid s_1 \leqslant s \leqslant s_2\}$ and let ∂G_2 be its complement in ∂G, i.e., $\partial G = \partial G_1 \cup \partial G_2$.

According to A. Douglis [3] we can consider the $2r$ elliptic equations in the normal form

$$\begin{gathered} u_{0_x} - v_{0_y} + p_0 u_0 + q_0 v_0 + h_0 = 0, \\ u_{0_y} + v_{0_x} + r_0 u_0 + s_0 v_0 + g_0 = 0, \\ u_{k_x} - v_{k_y} + a u_{k-1_x} + b u_{k-1_y} + \sum_{l=0}^{k} (p_l u_{k-l} + q_l v_{k-l}) + h_k = 0, \\ u_{k_y} + v_{k_x} + a v_{k-1_x} + b v_{k-1_y} + \sum_{l=0}^{k} (r_l y_{k-l} + s_l v_{k-l}) + g_k = 0, \\ k = 1, 2, \dots, r - 1, \end{gathered} \tag{1.5}$$

where h_k, g_k depend on u^{2r+1} and the functions d_i of (1.1).[1] Introducing complex coordinates $z = x + iy$, $\bar{z} = x - iy$ and

$$\begin{aligned} &w_k := u_k + iv_k, \qquad k = 0, 1, \dots, r-1, \\ &\omega := u^{2r+1}: \qquad \text{the vectors } \boldsymbol{\alpha} := (\alpha_0, \alpha_1, \dots, \alpha_{r-1})^T, \\ &\mathbf{w} := (w_0, w_1, \dots, w_{r-1})^T; \qquad \text{and } \gamma, \delta \text{ real functions,} \end{aligned} \tag{1.6}$$

[1] Without loss of generality we can assume that the first $2r$ equations in (1.1) in the functions $u^1, \dots, u^{2r}$ are elliptic.

the system (1.1) can be written in a brief form by using a hypercomplex algebra $\mathfrak{A}$ and hypercomplex functions in $\mathfrak{A}$:

$$w = \sum_{k=0}^{r-1} e^k w_k, \qquad e^r = 0;$$

however, for purposes of exposition we consider a somewhat simpler form having the *same principal part*, namely[2]

$$\begin{aligned} Dw + Aw + B\bar{w} + C\omega + \theta &= 0, \\ \omega_\lambda + \mathrm{Re}(\boldsymbol{\alpha} \cdot \mathbf{w}) + \gamma\omega + \delta &= 0, \end{aligned} \tag{1.7}$$

with

$$Dw := \left(\frac{\partial}{\partial \bar{z}} + q(z)\frac{\partial}{\partial z}\right) w. \tag{1.8}$$

$q(z) = \sum_{k=1}^{r-1} e^k q_k(z)$ is a nilpotent function; A, B, C, $\theta \in C_\alpha(\bar{G})$ are known hypercomplex functions; α, γ, $\delta \in C_\alpha(\bar{G})$ are known and ω_λ means the directional derivation of ω in the direction $\lambda = (\lambda_2, \lambda_1) \in C_\alpha{}^1(\bar{G})$ given by (1.4).

2. A System of Integral Equations

The first equation of (1.7) is of the form

$$\frac{\partial w_0}{\partial \bar{z}} = -a_0 w_0 - b_0 \bar{w}_0 - c_0\omega - \delta_0 =: \mathscr{L}_0(w, \omega)$$

$$\frac{\partial w_k}{\partial \bar{z}} = -\sum_{j=0}^{k-1} q_{k-j}\frac{\partial w_j}{\partial z} - \sum_{j=0}^{k} (a_{k-j}w_j + b_{k-j}\bar{w}_j) - c_k\omega - \delta_k =: \mathscr{L}_k(w, \omega)$$

$$k = 1, 2, \ldots, r-1. \tag{2.1}$$

Using the general representation theorem from Haack, Wendland [6], p. 259, Vidic [8], p. 15, we get

$$\begin{aligned} w_0(z, \bar{z}) = i \iint_G \Big\{ &\mathscr{L}_0(w, \omega)[G_z{}^{\mathrm{I}} + G_z^{\mathrm{II}}) + \overline{\mathscr{L}_0(w, \omega)}[G_{\bar{z}}{}^{\mathrm{I}} - G_{\bar{z}}^{\mathrm{II}})[d\zeta, d\bar{\zeta}] \\ &- \oint_{\partial G} \{u_0(d_n G^{\mathrm{I}} - i\, dG^{\mathrm{II}}) + iv_0\, d_n G^{\mathrm{II}}\Big\}, \end{aligned}$$

$$\begin{aligned} w_k(z, \bar{z}) = i \iint_G \Big\{ &\mathscr{L}_k(w, \omega)(G_z{}^{\mathrm{I}} + G_z^{\mathrm{II}}) + \overline{\mathscr{L}_k(w, \omega)}(G_{\bar{z}}{}^{\mathrm{I}} - G_{\bar{z}}^{\mathrm{II}})[d\zeta, d\bar{\zeta}] \\ &- \oint_{\partial G} \{u_k(d_n G^{\mathrm{I}} - i\, dG^{\mathrm{II}}) + iv_k\, d_n G^{\mathrm{II}}\Big\} \qquad k = 1, 2, \ldots, r-1. \end{aligned}$$

[2] See also, in this regard, Bojarski [2], Kühn [7], Gilbert and Wendland [5], Gilbert and Hile [4], and Begehr and Gilbert [1].

Here we are using the notation of Haack–Wendland [6], namely G^{I}, and G^{II} are the Green's function and the Neumann's function respectively for Laplace's equation. The normal derivative condition for G^{II} is defined as

$$d_n G^{\mathrm{II}} \mid_{\partial G} = -\Sigma^{-1}\sigma(s)\, ds,$$

where $\sigma(s)$ is any continuous function on ∂G such that $\int_{\partial G} \sigma(s)\, ds \neq 0$ (s is arc length) [see Haack and Wendland [6], p. 113.] The directional derivative d_n is usually defined with respect to the coefficients of a second order partial differential equation [6] (pp. 24–27). However, for our case of a complex system (2.1) which is component-wise in Hilbert normal form, we have

$$d_n \Phi := -i\Phi_z\, dz + i\Phi_{\bar{z}}\, d\bar{z}.$$

Introducing the hypercomplex functions

$$\begin{gathered} \hat{\mathscr{L}}(w, \omega) := \sum_{k=0}^{r-1} e^k \mathscr{L}_k(w, \omega) \\ u(z, \bar{z}) := \sum_{k=0}^{r-1} e^k u_k(z, \bar{z}), \qquad v(z, \bar{z}) := \sum_{k=0}^{r-1} e^k v_k(z, \bar{z}) \end{gathered} \tag{2.2}$$

we get by summing

$$\begin{aligned} w(z, \bar{z}) = i \iint_G \Big\{ & \hat{\mathscr{L}}(w, \omega)(G_z{}^{\mathrm{I}} + G_z^{\mathrm{II}}) + \overline{\hat{\mathscr{L}}(w, \omega)}(G_{\bar{z}}{}^{\mathrm{I}} - G_{\bar{z}}^{\mathrm{II}})[d\zeta, d\bar{\zeta}] \\ & - \oint_{\partial G} \{u(d_n G^{\mathrm{I}} - i\, dG^{\mathrm{II}}) + iv\, d_n G^{\mathrm{II}}\Big\}. \end{aligned} \tag{2.3}$$

If the contour integral is known, (2.3) is a system of Fredholm integral equations in which the function ω appears as a parameter. With $\lambda = (\lambda_1, \lambda_1) \in C_\alpha{}^1(\bar{G})$ given by (1.4) we introduce the Pfaffian forms

$$\Omega = \lambda_1\, dx + \lambda_2\, dy, \qquad \bar{\Omega} = \lambda_1\, dy - \lambda_2\, dx; \tag{2.4}$$

then the real characteristics of (1.1) are given by $\bar{\Omega} = 0$. Setting

$$\mathscr{L}_{2r+1}(w, \omega) := -\mathrm{Re}(\boldsymbol{\alpha} \cdot \mathbf{w}) - \gamma\omega - \delta \tag{2.5}$$

in the second equation of (1.7) and integrating we get

$$\omega(z, \bar{z}) = \omega(z', \bar{z}') + \int_{\substack{z', \bar{z}' \\ \bar{\Omega}=0}}^{z, \bar{z}} \mathscr{L}_{2r+1}(w, \omega)\, \Omega. \tag{2.6}$$

Introducing the linear space C^* of all continuous hypercomplex functions $\bar{G}$, we define the operators on C^* and $C_\alpha(\partial G)$, namely

$$\tilde{w}(z, \bar{z}) = \Psi(w) := i \iint_G \{w(G_z{}^{\mathrm{I}} + G_z^{\mathrm{II}}) + \bar{w}(G_{\bar{z}}{}^{\mathrm{I}} - G_{\bar{z}}^{\mathrm{II}})\}[d\zeta, d\bar{\zeta}],$$

$$F(z, \bar{z}) = \mathscr{C}(\phi) := -\oint_{\partial G} \phi(\zeta, \bar{\zeta})(d_n G^{\mathrm{I}} - i\, dG^{\mathrm{II}}), \tag{2.7}$$

$$\hat{w}(z, \bar{z}) = \chi(w) := \int_{\substack{z', \bar{z}' \\ \Omega = 0}}^{z, \bar{z}} w(\zeta, \bar{\zeta})\, \Omega.$$

Using this notation for solutions of generalized Cauchy–Riemann equations and the representation formulas of Haack and Wendland [6] p. 319, (13.1.2), we have

THEOREM 1. *Let the hypercomplex function* $w(z, \bar{z}) \in C_\alpha(\bar{G}) \cap C_\alpha{}^1(G)$, *and the function* $\omega(z, \bar{z}) \in C_\alpha(G)$, $\omega_\lambda \in C_\alpha(G)$ *be a solution pair of* (1.7); *then these functions solve the integral equations*

$$\begin{aligned} w &= \Psi(\mathscr{L}(w, \omega)) + \mathscr{C}(u) + iC_0 \\ \omega &= \chi(\mathscr{L}_{2r+1}(w, \omega)) + \omega', \end{aligned} \tag{2.8}$$

where

$$\omega' = \omega(z', \bar{z}'), \qquad u = \sum_{k=0}^{r-1} e^k u_k, \qquad v = \sum_{k=0}^{r-1} e^k v_k,$$

and

$$C_0 = N(v) = \frac{1}{\oint_{\partial G} \sigma(s)\, ds} \oint_{\partial G} v\sigma\, ds \tag{2.9}$$

is a constant. Conversely each solution pair (w, ω) *of* (2.8), (2.9) *with the above required differentiability solves the system* (1.7).

3. A SPECIAL BOUNDARY VALUE PROBLEM

Let ∂G_1 be the part of the boundary of G defined in Sect. 1. The system (1.7) in conjunction with the boundary conditions

$$\begin{aligned} u(z, \bar{z}) &= \phi(s) \in C_\alpha(0 \leqslant s \leqslant 1), \qquad z \in \dot{G}, \\ \omega(z, \bar{z}) &= \Psi(s) \in C_\alpha(s_1 \leqslant s \leqslant s_2), \qquad z \in \dot{G}_1, \end{aligned} \tag{3.1}$$

and the norm condition

$$\frac{1}{\oint_{\partial G} \sigma \, ds} \oint_{\partial G} v\sigma \, ds = N(v) = C_0 , \tag{3.2}$$

is called a *special boundary value problem* (SBVP).

We have the following

THEOREM 2. *If the coefficients of* (1.7)

$$A, B, C, \Theta, \alpha, \gamma, \delta \in C_\alpha(\bar{G}), \tag{3.3}$$

and the region G is sufficiently small (this condition is determined by the size of the coefficients (3.3)*), then*

(a) *There always exists a solution pair to the SBVP* (1.7), (3.1). *The components of this pair may be represented in the form,*

$$w = w_I + \kappa w_H , \qquad \omega = \omega_I + \kappa \omega_H . \tag{3.4}$$

where the hypercomplex function w_I, and the function ω_I are arbitrary solutions of the nonhomogeneous problem (1.7), (3.1), *and w_H, ω_H are arbitrary non identically vanishing solutions of the homogeneous problem* ($\Theta = \delta = \phi = \Psi = 0$); *$\kappa$ is a real parameter. Furthermore, the solution pair of the SBVP is unique if the norm condition is satisfied.*

(b) *For the solution pair* (3.4), *$w \in C_\alpha(\bar{G}) \cap C_\alpha{}^1(G)$, and for some β, $0 < \beta < \alpha$, $\omega \in C_i$ at each of the points $\gamma_G(s_1)$, $\gamma_G(s_2)$. Furthermore $\omega \in C_\alpha{}^1(G)$ along the curves $\bar{\Omega} = 0$ in G.*

Proof. Using Theorem 1 it is sufficient to prove the existence and uniqueness of the solution for the system of integral equations (2.8). As in Vidić [8] we consider with fixed C_0 the iterative scheme

$$\begin{aligned} w^n &:= \Psi(\hat{\mathscr{L}}(w^{n-1}, \omega^{n-1})) + \mathscr{C}(\phi) + iC_0 , \\ \omega^n &:= \chi(\mathscr{L}_{2r+1}(w^n, \omega^{n-1})) + \omega' \\ w^0 &= \omega^0 = 0, \qquad n = 1, 2,\ldots . \end{aligned} \tag{3.5}$$

If $w^{n*} := w^n - w^{n-1}$, $\omega^{n*} := \omega^n - \omega^{n-1}$ and $\hat{\mathscr{L}}_H(\cdot, \cdot)$ is the homogeneous part of $\hat{\mathscr{L}}(\cdot, \cdot)$ ($\Theta = 0$ in (1.7)) we have from (3.5)

$$\begin{aligned} w^{n*} &:= \Psi(\hat{\mathscr{L}}_H(w^{n-1*}, \omega^{n-1*})), \\ \omega^{n*} &:= \chi(\mathrm{Re}(\boldsymbol{\alpha} \cdot \mathbf{w}^{n*}) + \gamma\omega^{n-1*}). \end{aligned} \tag{3.6}$$

Introducing the norm $M_f := \sup_{z\in\bar{G}} \|f\| := \sup_{z\in\bar{G}} \sum_{k=0}^{r-1} |f_k|$ for a hypercomplex function f, it is not difficult to show that for suitable constants ψ_G, L_G, and M we have the inequalities

$$\| w^{n*} \| \leqslant K_G M((2r+1) M_{w^{n-1*}} + M_{\omega^{n-1*}}),$$

and (3.7)

$$| \omega^{n*} | \leqslant L_G M((2r+1) M_{w^{n*}} + M_{\omega^{n-1*}}).$$

The constants M, ψ_G, and L_G may be chosen as follows: Let $f(z)$ maps G onto the unit disk and $C := \max_{z,\zeta\in G} | f'(z)(z-\zeta)/f(z) - f(\zeta) |$, then

$$K_G := \frac{4C}{\pi} \max_{z\in G} \iint_G \frac{[d\zeta, d\bar{\zeta}]}{| z - \zeta |}.$$

The constant $M \geqslant \max\{| \alpha_k |, (k = 0,..., r-1); | \gamma |\}$, and L_G is the length of the longest characteristic $\bar{\Omega} = 0$ intersecting G.

Letting

$$\Delta_n := \{\max M_{w^{n*}}, M_{\omega^{n*}}\},$$

we obtain then

$$M_{w^{n*}} \leqslant (2r+1) MK_G \Delta_{n-1},$$

and

$$M_{\omega^{n*}} \leqslant ML_G((2r+1) MK_G + 1) \Delta_{n-1}.$$

Choosing q, such that $0 < q < 1$, and

$$K_G \leqslant \frac{q}{(2r+1) M}, \qquad L_G \leqslant \frac{q}{2M}, \tag{3.8}$$

it can be seen that the sequences $\{w_n\}$, $\{\omega_n\}$ converge uniformly to the function pair $(\tilde{w}, \tilde{\omega})$ which solve the SBVP (1.7), (3.1). The remaining parts of the proof including that concerning the qualitative behavior of ω at the points $\gamma_G(s_1)$, $\gamma_G(s_2)$ may be accomplished following arguments similar to that in Vidić [8].

4. Boundary Value Problems of Positive Index

In connection with the elliptic system (2.1) the boundary condition $u(z, \bar{z}) |_{\partial G} = \phi(s) \in C_\alpha(0 \leqslant s \leqslant 1)$ is a special case of the boundary conditions

$$\sum_{k=0}^{r-1} (a_k u_k + b_k v_k) e^k \Big|_{\partial G} = \sum_{k=0}^{r-1} \phi_k e^k \Big|_{\partial G} := \phi. \tag{4.1}$$

Setting $\gamma_k := a_k + ib_k$, (4.1) can be written as

$$\sum_{k=0}^{r=1} \operatorname{Re}(\bar{\gamma}_k w_k)\, e^k = \phi. \tag{4.2}$$

For fixed k, the index of γ_k is given by

$$n_k = \operatorname{Ind}(\bar{\gamma}_k) := \frac{1}{2\pi}\{\arg \bar{\gamma}_k(1) - \arg \bar{\gamma}_k(0)\}. \tag{4.3}$$

We consider the following boundary value problem:

$$Dw + Aw + B\overline{w} + C\omega + \Theta = 0, \tag{4.4}$$

$$\omega_\lambda + \operatorname{Re}(\boldsymbol{\alpha} \cdot \mathbf{w}) + \gamma\omega + \delta = 0, \tag{4.5}$$

$$\begin{gathered} \sum_{k=0}^{r-1} \operatorname{Re}(\bar{\gamma}_k w_k)\, e^k \Big|_{\partial G} = \phi \in C_\alpha(0 \leqslant s \leqslant 1), \\ |\bar{\gamma}_k| = 1, \qquad n_k = \operatorname{Ind}(\bar{\gamma}_k) > 0,\text{[3]} \qquad k = 0, 1, \ldots, r-1. \\ \omega\,|_{\partial G_1} = \Psi(s) \in C_\alpha(0 \leqslant s \leqslant 1). \end{gathered} \tag{4.6}$$

For fixed k each component of the elliptic system (4.4) is associated with an adjoint equation. Following Haack and Wendland [6] (p. 306), it is clear that for every solution pair (w, ω) such that $w \in C^0(\overline{G}) \cap C^1(G)$, $\omega \in C^0(\overline{G})$, and for each $\varkappa_k \in C^0(\overline{G}) \cap C^1(G)$, the integral relation

$$\begin{aligned} \oint_{\partial\Gamma} w_k \varkappa_k\, dz + \overline{w}_k \bar{\varkappa}_k\, d\bar{z} &= 2 \oint_{\partial\Gamma} \operatorname{Re}\{w_k \varkappa_k\, dz\} \\ &= \iint_\Gamma \{(w_k\varkappa_k)_{\bar{z}} - (\overline{w}_k \bar{\varkappa}_k)_z\}[d\bar{z}, dz] \\ &= 2i \iint_\Gamma \operatorname{Im}\{w_k(\varkappa_{k\bar{z}} - a_0\varkappa_k + \bar{b}_0\bar{\varkappa}_k)\}[d\bar{z}, dz] \\ &\quad + 4 \iint_\Gamma \operatorname{Im}\left\{\varkappa_k\left(\sum_{j=0}^{k-1}\left[q_{k-j}\frac{\partial w_j}{\partial z} + a_{k-j}w_j + b_{k-j}\overline{w}_j\right]\right.\right. \\ &\quad \left.\left. + c_k\omega + \delta_k\right)\right\} [dx, dy], \end{aligned} \tag{4.7}$$

holds for each Jordan domain $\Gamma \subset G$.

Therefore we have

[3] In the case that for some $k := (k_1 \cdots k_q)$ the indices $n_{k_1} = n_{k_2} = \cdots = n_{k_q} = 0$, see the remark after Theorem 6.

THEOREM 3. *For every solution pair* (w, ω) *of* (4.4), $w = \sum_{k=0}^{r-1} w_k e^k \in C^0(\bar{G}) \cap C^1(G)$, $\omega \in C^0(\bar{G})$, *and every solution* $z_k \in C^0(\bar{G}) \cap C^1(G)$, $(k = 0, 1, \ldots, r-1)$ *of the adjoint system*

$$\frac{\partial z_k}{\partial \bar{z}} = a_0 z_k - \bar{b}_0 \bar{z}_k , \tag{4.8}$$

the integral relation,

$$\oint_{\partial\Gamma} \operatorname{Re}\{w_k z_k\, dz\} = 2 \iint_{\Gamma} \operatorname{Im} \left\{ z_k \left(\sum_{j=0}^{k-1} \left[q_{k-j} \frac{\partial w_j}{\partial z} + a_{k-j} w_j + b_{k-j} \bar{w}_j \right] + c_k \omega + \delta_k \right) \right\} [dx, dy], \tag{4.9}$$

holds for each Jordan domain $\Gamma \subset G$.

The integral relation (4.9) leads directly to a correspondence between boundary conditions for the functions w_k and z_k. Assume $w(z, \bar{z})$ is a solution of (4.4) with the boundary conditions

$$\sum_{k=0}^{r-1} \operatorname{Re}(\bar{\gamma}_k w_k)\, e^k \Big|_{\partial G} = 0, \qquad \operatorname{Ind}(\bar{\gamma}_k) = n_k > 0, \qquad k = 0, 1, \ldots, r-1. \tag{4.10}$$

Then the function w_k satisfies

$$\operatorname{Im}(\bar{\gamma}_k w_k) \,|_{\partial G} = \rho_k(s) \neq 0,$$

and with $|\bar{\gamma}_k| = 1$,

$$w_k \,|_{\partial G} = i \frac{1}{\bar{\gamma}_k} \rho_k = i \gamma_k \rho_k(s).$$

If we write $dz \,|_{\partial G} = e^{i\theta(s)}\, ds$ on the boundary we have

$$\begin{aligned} \oint_{\partial G} \operatorname{Re}(w_k z_k\, dz) &= \oint_{\partial G} \operatorname{Re}(i \gamma_k \rho_k z_k e^{i\theta(s)})\, ds \\ &= \oint_{\partial G} \rho_k(s) \operatorname{Re}(i \gamma_k z_k e^{i\theta(s)})\, ds. \end{aligned} \tag{4.11}$$

Putting

$$\bar{\eta}_k = i \gamma_k e^{i\theta(s)} \tag{4.12}$$

it follows that

$$\operatorname{Ind}(\bar{\eta}_k) = \operatorname{Ind}(i \gamma_k e^{i\theta(s)}) = -n_k + 1 \leqslant 0, \tag{4.13}$$

where $\bar{\gamma}_k(s)$ and $\bar{\eta}_k(s)$ are said to be *mutually adjoint boundary conditions.* With these boundary data we have for the adjoint system (4.8) the following

THEOREM 4. ([6] p. 277) *The system*

$$\frac{\partial \varkappa_k}{\partial \bar{z}} = a_0 \varkappa_k - \bar{b}_0 \bar{\varkappa}_k \tag{4.8}$$

with the boundary conditions

$$\begin{gathered} \operatorname{Re}(\bar{\eta}_k \varkappa_k)|_{\partial G} = 0, \qquad \bar{\eta}_k = i\gamma_k e^{i\theta(s)}, \\ \operatorname{Ind}(\bar{\eta}_k) = -n_k + 1 \leqslant 0, \qquad k = 0, 1,\ldots, r-1 \end{gathered} \tag{4.14}$$

has for fixed k $2n_k - 1$ *non trivial, linearly independent continuous solutions* $\varkappa_k^{(\nu)}$, $\nu = 1, 2,\ldots, 2n_k - 1$.

With these solutions for the adjoint equation (4.8) in Theorem 4 we have for the left side of (4.9)

$$\begin{aligned} \oint_{\partial G} \operatorname{Re}(w_k \varkappa_k \, dz) &= \oint_{\partial G} \operatorname{Re}(w_k \bar{\gamma}_k \gamma_k \varkappa_k \, dz) \\ &= \oint_{\partial G} \{\operatorname{Re}(\bar{\gamma}_k w_k) \operatorname{Re}(\gamma_k \varkappa_k \, dz) - \operatorname{Im}(\bar{\gamma}_k w_k) \operatorname{Im}(\gamma_k \varkappa_k \, dz)\}. \end{aligned}$$

By construction we know that

$$\begin{aligned} \operatorname{Im}(\gamma_k \varkappa_k \, dz)|_{\partial G} &= \operatorname{Im}(\gamma_k \varkappa_k e^{i\theta(s)}) \, ds = -\operatorname{Im}(ii\gamma_k \varkappa_k e^{i\theta(s)}) \, ds \\ &= -\operatorname{Im}(i\bar{\eta}_k \varkappa_k) = -\operatorname{Re}(\bar{\eta}_k \varkappa_k)|_{\partial G} = 0, \end{aligned}$$

and

$$\begin{aligned} \operatorname{Re}(\gamma_k \varkappa_k \, dz) &= -\operatorname{Re}(ii\gamma_k \varkappa_k e^{i\theta(s)}) \, ds = -\operatorname{Re}(i\bar{\eta}_k \varkappa_k) \, ds \\ &= +\operatorname{Im}(\bar{\eta}_k \varkappa_k)|_{\partial G} \, ds. \end{aligned}$$

Setting

$$\rho_k^{(\nu)} = \operatorname{Im}(\bar{\eta}_k \varkappa_k^{(\nu)})|_{\partial G}, \tag{4.15}$$

we have

$$\oint_{\partial G} \operatorname{Re}(w_k \varkappa_k^{(\nu)} \, dz) = \oint_{\partial G} \phi k(s) \, \rho_k^{(\nu)} \, ds. \tag{4.16}$$

Therefore, we have from (4.9) the

THEOREM 5. *A necessary condition for the existence of a unique, continuous solution* (w, ω) *of* (4.4)–(4.6) *in G is that the integral conditions*

$$\oint_{\partial G} \phi_k(s)\, \rho_k^{(\nu)}(s)\, ds = 2 \iint_G \operatorname{Im} \left\{ \varkappa_k^{(\nu)} \left(\sum_{j=0}^{k-1} \left[q_{k-j} \frac{\partial w_j}{\partial \bar z} + a_{k-j} w_j + b_{k-j} \bar w_j \right] + c_k \omega + \delta_k \right) \right\} [dx, dy]$$
$$k = 0, 1, \ldots, r-1; \qquad \nu = 1, \ldots, 2n_k - 1, \tag{4.17}$$

hold.

In order to prove an existence theorem for the boundary value problem (4.4)–(4.6) we construct function sequences in the following way (see notation (2.1)) (see remark 2 after Theorem 6):

$$\begin{aligned} \frac{\partial w_0^{n+1}}{\partial \bar z} &= \mathscr{L}_0(w^{n+1}, \omega^n), \qquad \operatorname{Re}(\bar\gamma_0 w_0^{n+1})|_{\partial G} = \phi_0 , \\ \frac{\partial w_k^{n+1}}{\partial \bar z} &= \mathscr{L}_k(w^{n+1}, \omega^n), \qquad \operatorname{Re}(\bar\gamma_k w_k^{n+1})|_{\partial G} = \phi_k , \\ & \qquad\qquad k = 1, 2, \ldots, r-1; \end{aligned} \tag{4.18}$$

$$\omega^{n+1} = -\chi(\operatorname{Re}(\boldsymbol{\alpha} \cdot \mathbf{w}^{n+1}) + \gamma\omega^n + \delta) + \omega', \ \omega^{n+1}|_{\partial G} = \Psi$$
$$n = 0, 1, 2, \ldots, \quad \omega^0 = 0; \tag{4.19}$$

where the functions

$$w_0^{n+1}, w_1^{n+1}, \ldots, w_{r-1}^{n+1}, \omega^n, \quad n = 0, 1, \ldots$$

each fulfill the integral conditions (4.17). In order to show the convergence of the sequences (4.19) we introduce, as in the proof of Theorem 2, the functions

$$w_k^{n*} = w_k^{n} - w_k^{n-1}, \qquad \omega^{n*} = \omega^n - \omega^{n-1}. \tag{4.20}$$

From (4.18) we have

$$\begin{aligned} \frac{\partial w_0^{n+1*}}{\partial \bar z} &= -a_0 w_0^{n+1*} - b_0 \bar w_0^{n+1*} - c_0 \omega^{n*}; \qquad \operatorname{Re}(\bar\gamma_0 w_0^{n+1*})|_{\partial G} = 0, \\ \frac{\partial w_k^{n+1*}}{\partial \bar z} &= -a_0 w_k^{n+1*} - b_0 \bar w_k^{n+1*} \\ &\quad - \sum_{j=0}^{k-1} \left\{ q_{k-j} \frac{\partial w_j^{n+1*}}{\partial z} + a_{k-j} w_j^{n+1*} + b_{k-j} \bar w_j^{n+1*} \right\} - c_k \omega^{n*}, \end{aligned} \tag{4.21}$$
$$k = 1, 2, \ldots, r-1, \qquad \operatorname{Re}(\bar\gamma_k w_k^{n+1*})|_{\partial G} = 0;$$
$$\omega^{n+1*} = -(\operatorname{Re}(\boldsymbol{\alpha} \cdot \mathbf{w}^{n+1*}) + \gamma\omega^{n*}), \qquad \omega^{1*} = \omega^1.$$

To show the convergence of the sequences (4.19) we may again proceed as in Vidić [8], since the essential "Hilfssatz 5" in [8] p. 37 is also true for hypercomplex functions w when the integral conditions (4.17) hold.

We have then

THEOREM 6. *If for each* $n = 0, 1, 2,...$ *the functions*

$$w_0^{n+1}, w_1^{n+1},..., w_{r-1}^{n+1}, \omega^n, \quad n = 0, 1, 2,...$$

of the iteration scheme (4.18) *fulfills the integral conditions* (4.17), *then the boundary value problem* (4.4)–(4.6) *has a unique solution pair* (w, ω) *in a sufficiently small region* G. *Furthermore,* $w \in C_\alpha(\bar{G}) \cap C_\alpha^1(G)$, $\omega \in C_\alpha(G)$ *and* $\omega \in C_\beta$, $0 < \beta < \alpha$ *at each of the points* $\zeta_G(s_1)$ *and* $\zeta_G(s_1)$. *Along the curves* $\bar{\Omega} = 0$ *we have* $\omega \in C_\alpha^1$.

Remark 1. If for some $k := (k_1,..., k_q)$, the characteristics $n_{k_1} = \cdots = n_{k_q} = 0$ we have the same result as Theorem 6. For this $(k_1,..., k_q)$ we have neither integral conditions for the functions $w_{k_1},..., w_{k_q}$, nor the iteration functions $w_{k_1}^{n+1},..., w_{k_q}^{n+1}$.

Remark 2. Since it is always possible to reduce the boundary value problem associated with each component w_k to zero providing $|\gamma_k| = 1$ $(k = 0, 1,..., r-1)$ it is clear that one may solve the "reduced" equations (4.18) without recourse to checking whether the conditions (4.17) are satisfied.

5. AN EXAMPLE

In this section we apply the preceding results to the investigation of higher order, elliptic boundary value problems. In particular, we consider the Riquier problem

$$\Delta^2\Psi + a\,\Delta\Psi + b\Psi_x + c\Psi_y = f, \tag{5.1}$$

$$\begin{aligned} \Psi|_{\partial G} &= \tilde{\phi}_0(s) \in C_\alpha^1(\partial G), \\ \Delta\Psi|_{\partial G} &= \tilde{\phi}_1(s) \in C_\alpha^1(\partial G). \end{aligned} \tag{5.2}$$

Here the coefficients $a, b, c, f \in C_\alpha(\bar{G})$. By setting $\Phi := \Delta\Psi$, the equation (5.1) is transformed into the system

$$\begin{aligned} \Delta\Phi + a\Phi + b\Psi_x + c\Psi_y &= f, \\ \Delta\Psi - \Phi &= 0. \end{aligned} \tag{5.3}$$

Setting $v_0 := \Psi_x$, $u_0 := \Psi_y$, $v_1 := \Phi_x$, $u_1 := \Phi_y$, and identifying $\omega := \Phi$ yields a system of the type (1.5), namely

$$\begin{aligned} u_{0,x} - v_{0,y} &= 0 \\ u_{0,y} + v_{0,x} &= \omega \\ u_{1,x} - v_{1,y} &= 0 \\ u_{1,y} - v_{1,x} &= f - a\omega - bv_0 - cu_0 \\ \omega_x &= v_1 . \end{aligned} \tag{5.4}$$

Upon differentiating with respect to the arc length parameter, the boundary data may be rewritten as

$$\Psi_x \frac{dx}{ds} + \Psi_y \frac{dy}{ds} = \tilde{\phi}_0'(s) =: \phi_0(s) \quad \text{on } \partial G,$$

and

$$\Phi_x \frac{dx}{ds} + \Phi_y \frac{dy}{ds} = \tilde{\phi}_1'(s) =: \phi_1(s) \quad \text{on } \partial G.$$

Setting $\alpha := dy/ds$, and $\beta := dy/ds$ on ∂G these conditions take on the form

$$\begin{aligned} \alpha u_0 + \beta v_0 \,|_{\partial G} &= \phi_0(s) \in C_\alpha(\partial G), \\ \alpha u_1 + \beta v_1 \,|_{\partial G} &= \phi_1(s) \in C_\alpha(\partial G). \end{aligned}$$

The initial condition for the ω-unknown is

$$\omega \,|_{\partial G} = \psi(s) := \tilde{\phi}_1(s) \in C_\alpha^{\ 1}(\partial G). \tag{5.6}$$

Putting the system (5.4) into complex form yields a system of the type (2.1),

$$\begin{aligned} \frac{\partial w_0}{\partial \bar{z}} &= -c_0 \omega, \\ \frac{\partial w_1}{\partial \bar{z}} &= -a_1 w_0 - b_1 \overline{w}_0 - c_1 \omega - \delta_1 \\ \omega_x &= v_1 , \end{aligned}$$

where

$$a_0 = b_0 = \delta_0 = 0, \qquad q_1 = 0, \qquad c_0 = \frac{i}{2},$$

and

$$a_1 = \frac{1}{4}(b + ic), \qquad b_1 = -\frac{1}{4}(b - ic), \qquad c_1 = \frac{i}{2}a, \qquad \delta_1 = -\frac{i}{2}f.$$

In the complex notation the boundary conditions for w_k become $\operatorname{Re}(\bar{\gamma}_k w_k) = 0$ $(k = 0, 1)$, where $\gamma_k := \alpha + i\beta$. Since

$$\bar{\gamma}_k = \alpha - i\beta = \frac{dy}{ds} - i\,\frac{dx}{ds} = -ie^{i\theta(s)} \quad \text{on } \partial G,$$

it is clear that $\operatorname{ind}(\bar{\gamma}_k) = 1$.

The adjoint boundary value problem is then given by the system,

$$\frac{\partial \varkappa}{\partial \bar{z}}\, k = a_0 \varkappa_k - \bar{b}_0 \bar{\varkappa}_k = 0 \qquad (k = 0, 1), \tag{5.8}$$

with boundary conditions,

$$\operatorname{Re}(\bar{\eta}_k \varkappa_k)\,|_{\partial G} = 0, \qquad \bar{\eta}_k = i\gamma_k e^{i\theta(s)}, \qquad (k = 0, 1). \tag{5.9}$$

A simple computation shows that $\bar{\eta}_k = -1$, and hence, $\operatorname{ind}(\bar{\eta}_k) = -\eta_k + 1 = 0$ $(k = 0, 1)$. By Theorem (4), for each k we have then exactly one nontrivial, continuous solution, namely $\varkappa_k \equiv i\kappa$ $(k = 0, 1)$, where κ is a real constant.

We next investigate the integral conditions (4.17). For $k = 0$, the condition to be satisfied is

$$\oint_{\partial G} \Phi_0(s)\, \rho_0(s)\, ds = -\kappa \oint_{\partial G} \phi_0(s)\, ds = 2 \iint_G \operatorname{Im}\{\varkappa_0(c_0\omega + \delta_0)\, dx\, dy$$

$$= 2 \iint_G \operatorname{Im} \left\{ i\kappa \left(-\frac{i}{2}\, \omega \right) \right\} [dx, dy] = 0.$$

That the left-hand side also vanishes is seen quite easily by recalling the periodicity of $\tilde{\phi}_0(s)$, namely

$$-\kappa \oint_{\partial G} \phi_0(s)\, ds = -\kappa \int_{\partial G} \tilde{\phi}_0'(s)\, ds = -\kappa[\tilde{\phi}_0'(\ell) - \tilde{\phi}_0(0)] = 0.$$

We check next the integral conditions for $k = 1$. The right-hand side of (4.17) is seen to be

$$2 \iint_G \operatorname{Im}\{\varkappa_1(a_1 w_0 + b_1 \bar{w}_0 + c_1 \omega + \delta_1)\}\, dx\, dy$$

$$= 2 \iint_G \operatorname{Im} \left\{ i\kappa \left(\frac{i}{2}\, [bv_0 + cu_0] + \frac{ia}{2}\, \omega - \frac{i}{2} f \right) \right\} dx\, dy = 0.$$

Whereas the left-hand side is evaluated similarly as before to be

$$-\kappa \oint_{\partial G} \phi_1(s)\, ds = -\kappa \int_{\partial G} \tilde{\phi}_1'(s)\, ds = -\kappa(\tilde{\phi}_1(\ell) - \tilde{\phi}_1(0)) = 0.$$

Hence, since the integral conditions of Theorem (5) are valid there exists a unique, continuous solution of the system (5.4), (5.5), (5.6), and therefore also of the Riquier problem (5.1), (5.2).

REFERENCES

1. H. BEGEHR AND R. P. GILBERT, Randwertaufgaben ganzzahliger Charakteristik für verallgemeinerte hyperanalytische Funktionen, *Appl. Anal.* **6** (1977), 189–205.
2. B. B. BOJARSKI, The theory of generalized analytic vectors, *Ann. Polon. Math.* **17** (1966), 281–320.
3. A. DOUGLIS, A function-theoretic approach to elliptic systems of equations in two variables, *Comm. Pure Appl. Math.* **6** (1953), 259–289.
4. R. P. GILBERT AND G. N. HILE, Generalized hypercomplex function theory, *Trans. Amer. Math. Soc.* **195** (1974), 1–29.
5. R. P. GILBERT AND W. WENDLAND, Analytic, generalized, hyperanalytic function theory and an application to elasticity, *Proc. Royal Soc. Edinburgh Ser. A* **73**, No. 22 (1975), 317–331.
6. W. HAACK AND W. WENDLAND, "Vorlesungen über Partielle und Pfaffsche Differentialgleichungen," Birkhäuser, Basel/Stuttgart, 1969.
7. E. KÜHN, "Über die Funktionentheorie und das Ähnlichkeitsprinzip einer Klasse elliptischer Differentialgleichungssysteme in der Ebene," Dissertation, Dortmund, 1974.
8. CH. VIDIC, "Über zusammengesetzte Systeme partieller linearer Differentialgleichungen erster Ordnung," Dissertation D 83, Technische Universität Berlin, 1969, 45 S.